Comment on "Ising model on a small world network"

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In the recent study of the Ising model on a small-world network by A. Pękalski [Phys. Rev. E **64**, 057104 (2001)], a surprisingly small value of the critical exponent $\beta \approx 0.0001$ has been obtained for the temperature dependence of the magnetization. We perform extensive Monte Carlo simulations of the same model and conclude, via the standard finite-size scaling of various quantities, that the phase transition in the model is of the mean-field nature, in contrast to the work by A. Pękalski but in accord with other existing studies.

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A. Pękalski in Ref. [1] studied the Ising model on a small-world network constructed from a ring lattice. In the presence of a finite fraction of additional long-range interactions, the model was observed to undergo a phase transition at a finite temperature, in agreement with other related studies [2, 3, 4]. However, the exponent β, describing the critical behavior of the magnetization in the vicinity of the transition, was found to be very small: $\beta \approx 0.0001$, in contrast to the previous studies suggesting the mean-field nature of the transition [2, 3, 4]. The small-world network in Ref. 1 was constructed in a slightly different way compared with the original model by Watts and Strogatz [5], under the additional constraint that not more than one shortcut is allowed for each vertex in the network. In this comment we present results of extensive Monte Carlo (MC) simulations of the same model as that in Ref. 1, which reveal that the phase transition is described by the mean-field exponents, $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, and $\bar{\nu} = 2$ (see below for definitions), and thus conclude that the additional constraint in the network construction does not change the universality class of the transition.

For simplicity, we consider only the network where every vertex has one shortcut (or every spin has three couplings), which is identical to Pękalski's network with the parameter $p = 1$ in Ref. 1. We then perform extensive Monte Carlo simulations of the Ising model described by the Hamiltonian:

$$H = -\frac{J}{2} \sum_i \sum_{j \in \Lambda_i} \sigma_i \sigma_j, \qquad (1)$$

where J is the coupling strength, $\sigma_i (= \pm 1)$ is the Ising spin on vertex i, and Λ_i denotes the neighborhood of vertex i, including those vertices connected to i. For given network size N at temperature T (in units of J/k_B), we have measured various thermodynamic quantities such as Binder's cumulant [7], the specific heat, and the susceptibility:

$$U_N = 1 - \frac{[\langle m^4 \rangle]}{3[\langle m^2 \rangle]^2} \qquad (2)$$

$$C_v = \frac{[\langle H^2 \rangle - \langle H \rangle^2]}{T^2 N} \qquad (3)$$

$$\chi = \frac{1}{N} \sum_{ij} [\langle \sigma_i \sigma_j \rangle] \qquad (4)$$

with $m \equiv |(1/N) \sum_i \sigma_i|$, where $\langle \cdots \rangle$ and $[\cdots]$ represent the thermal average (taken over 5000 MC steps after discarding 5000 MC steps for equilibration at each temperature) and the average over different network realizations (taken over 400-1200 different configurations), respectively.

Binder's cumulant in Eq. (2), plotted for various sizes, yields a unique crossing point as a function of the temperature T, providing a convenient method to determine the critical temperature T_c. Figure 1(a) shows that the result $T_c = 1.81(2)$ is obtained from the crossing point of Binder's cumulant for $N \geq 800$. The critical exponent $\bar{\nu}$, describing the divergence of the correlation volume in such a way that $\xi_V \sim |T - T_c|^{-\bar{\nu}}$ [4], can be determined from the expansion of U_N near T_c [6]:

$$\Delta U_N \equiv U_N(T_1) - U_N(T_2) \propto N^{1/\bar{\nu}}, \qquad (5)$$

where T_1 and $T_2 (> T_1)$ are chosen near T_c. Figure 1(b) results in the value $\bar{\nu} \approx 2.0$, which, together with the hyperscaling relation $\bar{\nu} = 2 - \alpha$, gives the critical exponent $\alpha \approx 0$ for the specific heat C_v.

With such a mean-field value, we write the finite-size

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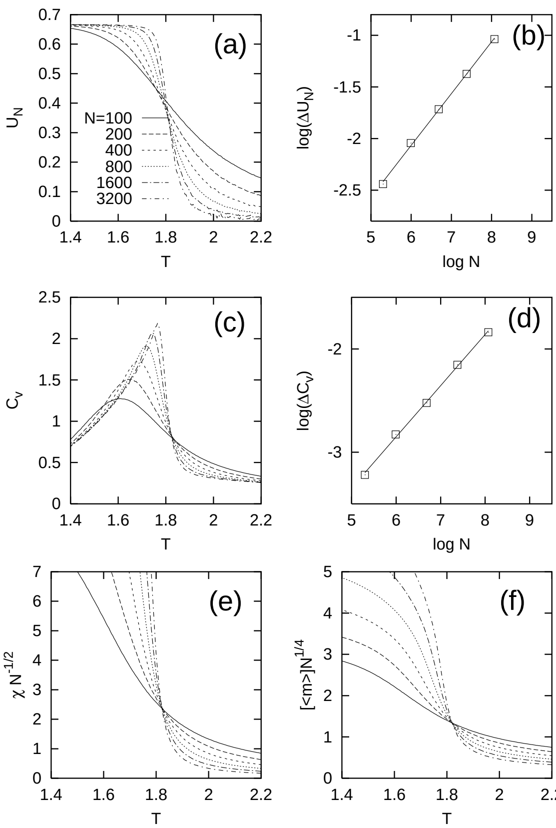

FIG. 1: (a) Binder's cumulant U_N has a unique crossing point at $T_c = 1.81(2)$ (in units of J/k_B). (b) The critical exponent $\bar{\nu} = 1.99$ is obtained from the least-square fit to the form in Eq. (5). (c) Specific heat C_v (in units of k_B) also has a crossing point at $T_c = 1.82(2)$, suggesting $\alpha = 0$. (d) From the similar expansion of C_v near T_c, $\bar{\nu} = 2.02$ is obtained. (e) Finite-size scaling of the susceptibility again determines $T_c = 1.82(2)$ with the critical exponent $\gamma = 1$. (f) Finite-size scaling of the magnetization $[\langle m\rangle]$ [see Eq. (8)] with $\beta = 1/2$ also leads to the crossing point at $T_c = 1.82(1)$. In (a), (c), (e), and (f), simulations have been performed with the temperature increment $\Delta T = 0.005$ whereas the data for $N \geq 200$ have been used for fitting in (b) and (d).

scaling in the form

$$C_v = f\Big((T - T_c)N^{1/\bar{\nu}}\Big), \tag{6}$$

where $f(x)$ is an appropriate scaling function with the scaling variable x. As shown in Fig. 1(c), the unique crossing point of C_v again yields $T_c = 1.82(2)$, in accord with T_c obtained from U_N within numerical errors. The similar expansion of C_v then provides an alternative way of determining $\bar{\nu}$: $\Delta C_v = C_v(T_1) - C_v(T_2) \propto N^{1/\bar{\nu}}$, leading to the estimation $\bar{\nu} \approx 2.0$ in Fig. 1(d).

The divergence of the susceptibility when T_c is approached from above is described by the critical exponent γ: $\chi \sim (T - T_c)^{-\gamma}$, which suggests the finite-size scaling form

$$\chi = N^{\gamma/\bar{\nu}} g\Big((T - T_c)N^{1/\bar{\nu}}\Big) \tag{7}$$

with the appropriate scaling function $g(x)$. Combined with $\bar{\nu} \approx 2$ found above, the finite-size scaling form (7) yields the value $\gamma \approx 1$ and $T_c = 1.82(2)$ as shown in Fig. 1(e).

Finally, on the basis of the above observation, the critical exponent β for the magnetization is then determined from the scaling form

$$[\langle m\rangle] = N^{-\beta/\bar{\nu}} h\Big((T - T_c)N^{1/\bar{\nu}}\Big), \tag{8}$$

which leads to $T_c = 1.82(1)$ and $\beta \approx 1/2$ [see Fig. 1(f)].

We note that the obtained critical temperature appears to be higher by factor two than that in Ref. 1. The standard mean-field approximation applied to this model, where the coordination number is three, yields $T_{MF} = 3$ [8]. Considering the infinite range of the interactions and the mean-field nature of the transition, we believe that our estimation $T_c = 1.82(2)$ (in units of J/k_B) is more precise. In this respect, it is interesting to note that also in the XY model on the small-world network a relatively large value $T_c/T_{MF} \approx 0.9$ has been estimated [4].

In summary, we have numerically studied the Ising model on the small-world network constructed in the identical way as in Ref. 1. In contrast to Ref. 1, we have obtained the standard mean-field critical exponents: $\alpha = 0, \beta = 1/2, \gamma = 1$, and $\bar{\nu} = 2$ and confirmed that the phase transition is of the mean-field nature, in agreement with other previous studies [2, 3, 4].

[1] A. Pękalski, Phys. Rev. E **64**, 057104 (2001).
[2] M. Gitterman, J. Phys. A: Math. Gen. **33**, 8373 (2000).
[3] A. Barrat and M. Weigt, Eur. Phys. J. B **13**, 547 (2000);
[4] B.J. Kim, H. Hong, P. Holme, G.S. Jeon, P. Minnhagen, M.Y. Choi, Phys. Rev. E **64**, 056135 (2001).
[5] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[6] B.J. Kim, P. Minnhagen, S.K. Oh, and J.S. Chung, Phys. Rev. B **64**, 024406 (2001).
[7] K. Binder and D.W. Heerman, *Monte Carlo Simulation in Statistical Physics*, 2nd ed. (Springer-Verlag, Berlin, 1992).
[8] See, e.g., N. Goldenfeld, *Lectures on phase transitions and the renormalization group* (Addison Wesley, 1992).